THE
AMERICAN
MATHEMATICAL MONTHLY.

Entered at the Post-office at Springfield, Missouri, as second-class matter.

VOL. XI. JUNE-JULY, 1904. NO. 6-7.

SPHERICAL GEOMETRY.

By EDWIN BIDWELL WILSON.

LECTURE VI. RELATIONS OF SYMMETRY.

Definition: *Two triangles ABC and $A'B'C'$ are said to be symmetrical when their corresponding sides are congruent and their corresponding angles are symmetrical.* That is, when the following congruences are satisfied

$$AB \equiv A'B', \qquad BC \equiv B'C', \qquad CA \equiv C'A',$$
$$\measuredangle BCA \equiv \measuredangle A'C'B', \quad \measuredangle CAB \equiv \measuredangle B'A'C', \quad \measuredangle ABC \equiv \measuredangle C'B'A',$$

Theorem 41. *Two triangles symmetrical to the same triangle or to congruent triangles are congruent to each other. If two triangles are respectively congruent and symmetrical to the same triangle or to congruent triangles they are symmetrical to each other.*

Theorem 42. *Two triangles ABC and $A'B'C'$ which have two sides of the one congruent to two sides of the other but the included angles symmetrical are symmetrical.*

Theorem 43. *Two triangles which have a side of the one congruent to the side of the other but the adjacent angles respectively symmetrical are symmetrical.*

Theorem 44. *A triangle which has two congruent sides or two symmetrical angles is symmetrical to itself.*

The first of these theorems follows directly from the congruences which by definition hold between symmetrical and between congruent triangles. To prove the second let $A^\circ B^\circ C^\circ$ be any triangle symmetrical to the given triangle ABC and let $A'B'C'$ be connected with ABC by the congruences

	$AB \equiv A'B'$,	$BC \equiv B'C'$,	$ABC \equiv C'B'A'$.	Hyp.
Then	$A^\circ B^\circ \equiv AB$,	$B^\circ C^\circ \equiv BC$,	$\measuredangle C^\circ B^\circ A^\circ = \measuredangle ABC$.	Def.
Hence	$A^\circ B^\circ \equiv A'B'$,	$B^\circ C^\circ \equiv B'C'$,	$\measuredangle C^\circ B^\circ A^\circ \equiv \measuredangle C'B'A'$.	Ax.V

Hence the triangles $A'B'C'$ and $A^\circ B^\circ C^\circ$ are congruent and Theorem 41 may be adduced to show that the triangles ABC and $A'B'C'$ are symmetrical. Theorem 43 may be demonstrated in a similar manner and the last of the theorems is evidently nothing but a corollary of the two which precede it.

There is, however, in this proof one very serious assumption which must be taken as a further axiom unless it can be proved. We said "let $A^\circ B^\circ C^\circ$ be any triangle symmetrical to the given triangle ABC" and the fact that there could be found at least one triangle symmetrical to the given triangle was an essential element in the proof. Now it is perfectly conceivable that there should be no two triangles which satisfy the conditions imposed on symmetrical triangles. The definition states that the corresponding sides must be congruent and the corresponding angles symmetrical. Here are six conditions imposed on the triangle symmetrical to a given triangle, whereas any triangle is completely determined by three conditions. It may well be that there are no triangles which satisfy such conditions. In order to make the proof valid it is necessary to show the existence of at least one triangle which shall be symmetrical to any given triangle.* This will follow easily (see Theorem 57) after the establishment of the next few theorems.

Theorem 45. *If two lines are perpendicular at one point of intersection O, they are perpendicular at the antipodal point O'.*

Let a and a' be the opposite directions on one of the lines and let b be one of the directions along the other. Let p, p', q be the directions which issue from O' along the semi-lines defined by a, a', b, respectively. Perform the motion which carries a into b. The angle (a, b) will be carried into its congruent (b, a'). The point O' remains at rest (Theorem 28) and the directions p, q are carried into q, p', respectively (Axiom V). Hence the angles (p, q) and (q, p') are right angles.

*This is not the first theorem for the existence of some geometric object which we have had to prove. Theorem 2 showed the existence of the inverse rigid motion when any particular motion was in question. Theorem 4 was given for the purpose of enabling us to affirm the existence of a limiting point of a certain series. Lecture IV was to a large extent concerned with establishing the existence of a natural order among the directions issuing from a point. Theorem 35 had to do with the existence of a bisector of any given angle. Incidentally one may note that the proof of the existence of some object need not give any clue as to how that object may be obtained or constructed. In fact, there are two radically different methods of procedure in establishing existence: in one the object desired is actually found or constructed and thus its existence is assured, in the other nothing is shown save that the desired object can be found if looked for in the right place. Of course the former method is in many ways more satisfactory, but for the pure logical development of a subject the latter is about as useful and often it is easier. It is probably no exaggeration to say that one of the most important changes which the last half century has produced in mathematical thought is the passing of the old attitude of facile assumption in matters regarding the existence of the objects defined and the advent of a new attitude of rigorous insistence on the proofs of this existence. This change was a necessary precursor to anything like true logical precision and as such it was invaluable. It is, however, easily carried to extremes, and with deleterious effects on scientific instruction and investigation; for, as was pointed out in the last Lecture, a too constant insistence on precision renders the intuition numb and turns originality into timidity.

Theorem 46. *A line drawn on a hemisphere perpendicular to the line which bounds the hemisphere divides it into two congruent parts.*

Theorem 47. *Two perpendiculars erected to the boundary of a hemisphere at points subtending a quadrantal segment divide the hemisphere into four congruent tri-rectangular tri-quadrantal triangles and divide the entire surface into eight such triangles.*

Of these theorems the first follows as a corollary of Theorems 45 and 27; the second contains the term quadrantal segment which must be defined and justified. By Theorem 5 is established the fact that any segment AB mey be carried into the segment BA by a rotation about some point C such that AC is congruent to BO. The point C may be called the *middle* point of the segment AB. Moreover as BC must in a similar manner be congruent to CB, it follows that AC is congruent to CB. Hence the point C satisfies a definition for bisector of a segment similar to that given in Theorem 35 for the bisector of an angle. *The middle point C of a segment AB is a point such that AC is congruent to CB.* Application of this to the case of a semi-line OO' shows that there is a point C such that OC is congruent to CO'. Such segments may be called quadrantal segments or quadrants. It will be left to the reader to prove the evident theorem that all quadrantal segments are congruent.

Let the line l be the boundary of the hemisphere considered. Let A and B be two points such that AB is a quadrantal segment. Further, let the perpendiculars erected at A and B cut in O and cut the line l in A' and B', respectively. The segments BA', $A'B'$, and $B'A$ are quadrants as may be seen by applying to the semi-lines intercepted between A, A' and B, B' the theorem that all quadrants are congruent. Hence the triangles ABO, $BA'O$, $A'B'O$ are mutually congruent (Theorems 39 and 11). Hence the four corresponding angles at O are mutually congruent and therefore right angles. The triangles are tri-rectangular. Also the corresponding sides AO, BO, $A'O$, $B'O$ are congruent and consequently are quadrants. The hemisphere has been divided into four congruent tri-rectangular triangles of which each side is a quadrantal segment. That all such triangles are congruent and that there are eight formed on the surface of the sphere by three mutually perpendicular lines is left to the reader to prove.

Theorem 48. *If on a given line l a point A is moved into the antipodal point A' and the direction a, issuing from A and lying along l, is moved into the direction a', issuing from A and lying along l in the same order, the motion is involutory and each point of the line l is moved into its antipodal point.*

The proof of this theorem is similar to that of Theorem 34. Let B be any point of l and B' the antipodal point. Suppose that B were carried into B°, and B' into B'°. The motion is involutory: for repeating it brings A' and a' back to A and a and hence all points return to their original positions. If $\mathbf{M}$ denote the motion, we have

$$A,\ A',\ B,\ B',\ B^{\circ},\ B'^{\circ},\ [\mathbf{M}]\ A',\ A,\ B^{\circ},\ B'^{\circ},\ B,\ B'.$$

The order of the points ABB'° is the same as that of $A'B^{\circ}B'$. Hence both the

points B°, B'° lie in the same semi-line BB' and not in different semi-lines. But $B^\circ B'^\circ$ is derived from BB' by a motion and must be a semi-line. Hence the contradiction — unless B' and B° coincide. The theorem is proved.

Theorem 49. *All the perpendicular erected to a given line meet in a pair of antipodal points.*

Let l be the given line, AA' and BB' two perpendiculars to it intersecting in the points O, O'. Let CC' be any other perpendicular. Consider the hemisphere containing O. Move C into C' and the direction CA in the direction $C'A'$. By Theorem 46 the portion of the surface between the perpendicular CC' and one half of the line l will be carried into the region bounded by CC' and the other half of the line l. Hence if O does not lie on CC' it will have passed from one side of it to the other. But by Theorem 48 the points A and A' will have interchanged their positions. The right angle at A will have taken the place of the congruent right angle at A' and vice versa. The same interchange will have taken effect in the case of B and B'. Hence the perpendiculars AA' and BB' will have remained unchanged as a whole and the point O cannot have passed from one side of CC' to the other. Hence the point O lies on the line CC', which was any perpendicular to the line l, and the theorem is proved.

Definition: *The points in which the perpendiculars to a given line meet are called the poles of that line.*

Theorem 50. *Every line drawn through the pole of a given line cuts the given line at right angles.*

Theorem 51. *If a quadrantal segment OA rotates about the point O the extremity A describes a line to which the rotating segment is always perpendicular.*

Definition: *The line described by the moving end of a quadrant which rotates about one end O is called the polar of the point O.*

Theorem 52. *If two semi-lines OAO' and OBO' meet in the two points O and O', the angles formed at O and O' are congruent if the corresponding sides lie on different semi-lines, symmetrical if they lie on the same semi-lines.*

Theorem 53. *From any point without a given line, other than the pole of the line, one and only one line can be drawn perpendicular to the given line.*

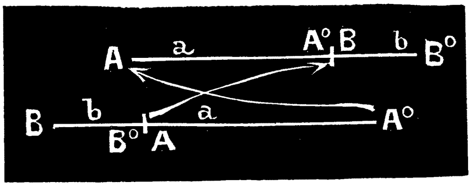
Theorem 54. *Two right triangles which have the hypothenuse and an acute angle of the one congruent to the hypothenuse and an acute angle of the other are congruent.*

Of these five theorems the first is an immediate corollary of Theorem 49, and the second may be proved from these two by showing that the line passed through any two positions of the point A is the required locus and satisfies the stated conditions. Theorem 52 is readily demonstrated by drawing the polar line of O, thus obtaining two bi-rectangular triangles with a common base (apply Theorems 5 and 10 or 11). To find one perpendicular to a given line and passing through a given point draw the line connecting the point to the pole of the line. There cannot be more than one, because two perpendiculars to a line intersect only in the poles of the line. The last theorem is proved by applying the congruent hypothenuses and allowing the congruent angles to coincide. Theorem 53 will then establish the uniqueness of the other side.

To take up the proof that to any triangle there exists a symmetrical triangle we shall prove the following lemma, which will be so important when we come to discuss the idea of measure that we may denote it as a theorem and refer back to it at that time.

Definition of the sum of two segments: *Given the two segments AA°, represented by a, and BB°, represented as b. The sum $a+b$ is the segment AB° obtained by laying BB° down along the line AA° produced so that B, the initial point of the second segment, coincides with A°, the final point of the first segment.*

Theorem 55. *The sum, $a+b$, of two segments a and b is congruent to the sum, $b+a$, of the segments taken in inverse order.*



Let a be designated as AA°, and b as BB°. Then $a+b$ is $A(A^\circ B)B^\circ$ where A° and B are coincident points. In like manner $b+a$ is $B(B^\circ A)A^\circ$ where B° and A coincide. Apply A° of the second to A of the first in such a way that the direction from A° to A falls along the direction from A to A° in the first figure. From Theorem 5 it follows that A falls on A°. Then $B^\circ B$ of the second figure will take the direction BB° of the first and B will fall on B°. The segments AB° and $A^\circ B$ coincide and are therefore congruent. Hence AB° and BA° are congruent and the theorem is proved.

Theorem 56. *If two segments c and d be respectively added to a given segment x and yield congruent segments $c+x$ and $d+x$, the segments c and d are congruent.*

In case they are added on the same end of the segment x the theorem is evident at once. In case they are added on opposite ends a proof similar to that given above and dependent on Theorem 5 may be given.

Theorem 57. *To a given triangle there may be found a symmetrical triangle.*

Let ABC be the given triangle. Produce the sides AB and AC until they meet in the point A' antipodal to A. In like manner produce the other pairs of sides and obtain the points B' and C'. The triangles ABC and $A'B'C'$ are symmetrical. For in the first place AA' and BB' are semi-lines with the common segment $B'A$. Hence by Theorem 56 the segments AB and $A'B'$ respectively which added to the same segment $B'A$ yield the congruent semi-lines, are themselves congruent. In like manner BC and CA are congruent to $B'C'$ and $C'A'$. And in the second place the angle BAC at A is either congruent or symmetrical to the angle $B'A'C'$ at A' (Theorems 34 and 52). Now AB and $A'B'$ are directed in the same order along the line on which they lie. C and C' lie on opposite sides of this line. Hence if AB be moved into coincidence with $A'B'$ the sides AC will not take the direction $A'C'$ but will lie on the other hemisphere from it. Consequently the angles are not congruent and must be symmetrical. In like manner the other corresponding angles may be proved to be symmetrical. The triangle $A'B'C'$ therefore satisfies the six conditions of congruent sides and symmetrical angles imposed in the definition given at the beginning of the lecture. The theorems concerning symmetrical triangles are valid and the development of such theorems as the following may be left to the reader with merely a caution

generally to substitute "symmetrical" for the "equal" which is found in most books or for the "congruent" found in some.

Theorem 58. *The bisector of the vertical angle of an isosceles triangle, the perpendicular bisector of the base, and the perpendicular let fall from the vertex to the base are all coincident.*

Theorem 59. *In a triangle symmetrical angles are opposite congruent sides and conversely. In case the angles are not symmetrical the greater angle is opposite the greater side and conversely.*

Theorem 60. *Of oblique lines drawn from a point to a line those cutting off the greater segments from the foot of the perpendicular are the greater or the less according as the perpendicular is less or greater than a quadrant.*

To quote the theorems concerning the locus of points equidistant from two given points or from two given lines and others like them would be merely tedious. Enough has been said to show the great importance of the ideas of symmetry to the development of spherical geometry and of plane geometry in case all operations are to be restricted to a plane. The really instructive part of the work was the necessity of establishing the existence of such objects as symmetrical triangles. It may be noted that as yet no construction for the middle point of a line or for the bisector of an angle or for a right angle has been given. These come in more appropriately under the topic of circles. The knowledge of the existence of the elements desired and not a method of finding them has been all that has proved necessary. It may also be noted that no measure either of segments or of angles has been introduced. The ideas of congruence, of "greater than" and "less than" which were derived from congruence, and even of the sum of two segments could all be carried out in a purely geometrical way by aid of Axiom V with no assistance from the theory of number.

A SIMPLE CONSTRUCTION FOR FINDING THE DIAMETER OF A GIVEN MATERIAL SPHERE.

By H. S. QUACKENBUSH, Lawrence Scientific School, Harvard University.

The following construction for finding the diameter of a given material sphere appears simpler and more direct than the one given in Wentworth's, Phillips and Fisher's, and other current text-books.

From any two points P and Q on the sphere as poles, with compass-opening a, describe two arcs on the sphere, intersecting at A and B. From the same two points as poles, with a different compass-opening b, describe two more arcs, intersecting at C and D. These four points, A, B, C, D, will then lie on a great circle of the sphere, namely, the perpendicular bisector of the arc PQ.

To find the diameter of the sphere we have then simply to choose any three of these four points, say A, B, C, measure with the compasses the straight line distances between them, lay off the plane triangle ABC, and find the diameter of its circumscribed circle.

This construction, like the one usually given, requires only a small part of the spherical surface to be accessible. A good check on the accuracy of the work can be obtained by making a different choice of the three points.